Correction DS10.

Préambule : $M_\alpha = M_B(f) = \begin{pmatrix} 2\alpha+1 & -\alpha & \alpha+1 \\ \alpha-2 & \alpha-1 & \alpha-2 \\ 2\alpha-1 & \alpha-1 & 2\alpha-1 \end{pmatrix}$

Partie A : $\alpha = -1$

1. $M_{-1} = \begin{pmatrix} -1 & 1 & 0 \\ -3 & -2 & -3 \\ -3 & -2 & -3 \end{pmatrix}$ $\quad rg(M_{-1}) = ?$

$L_3 - L_2 \to L_3$ $\begin{pmatrix} -1 & 1 & 0 \\ -3 & -2 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad L_2 - 3L_1 \to L_2$ $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -5 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow rg(M_{-1}) = 2.$

2. $v_1 \in \ker f \iff f(v_1) = 0_{\mathbb{R}^3}$. Or $f(v_1) = f(3e_1 + 3e_2 - 5e_3)$

$= 3f(e_1) + 3f(e_2) - 5f(e_3) = -3e_1 - 9e_2 - 9e_3 + 3e_1 - 6e_2 - 6e_3 + 15e_2 + 15e_3$

$= 0_{\mathbb{R}^3}$ cqfd.

On sait $rg\, f = rg(M_{-1}) = 2 \Rightarrow \dim \ker f = 1$ d'après thm du rg car $\dim \mathbb{R}^3 = 1$. Or $v_1 \in \ker f$ donc $\{v_1\}$ base de $\ker f$.

3. $f(u) = v_1$. Posons $U = M_B(u) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad V_1 = M_B(v_1) = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}$

alors $f(u) = v_1 \iff M_{-1} U = V_1 \iff \begin{cases} -x + y = 3 \\ -3x - 2y - 3z = 3 \\ -3x - 2y - 3z = -5 \end{cases} \Rightarrow$ absurde

donc $\nexists$ pas de u tq $f(u) = v_1$.

cela signifie donc que $v_1 \notin \operatorname{Im} f$. donc $\ker f \cap \operatorname{Im} f = \{0_{\mathbb{R}^3}\}$ et on a : $\dim \ker f + \dim \operatorname{Im} f = 3 = \dim \mathbb{R}^3$ donc

$\mathbb{R}^3 = \ker f \oplus \operatorname{Im} f.$

4. $\alpha v_1 + \beta v_2 + \gamma v_3 = 0 \Rightarrow \alpha = \beta = \gamma = 0$ (trivial) $\Rightarrow B'$ base de $\mathbb{R}^3$ car $\operatorname{card} B' = \dim \mathbb{R}^3$

$f(v_1) = 0_{\mathbb{R}^3}$ car $v_1 \in \ker f$.

$f(v_2) = -f(e_1) + f(e_2) + f(e_3) = 2(-e_1 + e_2 + e_3) = -2v_2$

$f(v_3) = f(e_1) - 3f(e_2) - 3f(e_3) = -4(e_1 - 3e_2 - 3e_3) = -4v_3$

$\Rightarrow D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix} = M_{B'}(f)$

5/ $P = P_{\mathcal{B}\to\mathcal{B}'} = \begin{pmatrix} 3 & -1 & 1 \\ 3 & 1 & -3 \\ -5 & 1 & -3 \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{8}\begin{pmatrix} 0 & 1 & -1 \\ -12 & 2 & -6 \\ -4 & -1 & -3 \end{pmatrix}$

6. $\forall n \in \mathbb{N}^*$ $D^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-2)^n & 0 \\ 0 & 0 & (-4)^n \end{pmatrix}$ et $M^n = P D^n P^{-1}$.

7. $M^{2009} = \frac{1}{8}\begin{pmatrix} 3 & -1 & 1 \\ 3 & 1 & -3 \\ -5 & 1 & -3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & (-2)^{2009} & 0 \\ 0 & 0 & (-4)^{2009} \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ -12 & 2 & -6 \\ -4 & -1 & -3 \end{pmatrix}$

$= \frac{(-2)^{2009}}{8}\begin{pmatrix} 3 & -1 & 1 \\ 3 & 1 & -3 \\ -5 & 1 & -3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{2009} \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ -12 & 2 & -6 \\ -4 & -1 & -3 \end{pmatrix}$

$= \frac{(-2)^{2009}}{8}\begin{pmatrix} 0 & -1 & 2^{2009} \\ 0 & 1 & -3.2^{2009} \\ 0 & 1 & -3.2^{2009} \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ -12 & 2 & -6 \\ -4 & -1 & -3 \end{pmatrix}$

$= (-1)^{2009}\, 2^{2006}\begin{pmatrix} 12-4.2^{2009} & -2-2^{2009} & 6-3.2^{2009} \\ -12+12.2^{2009} & 2+3.2^{2009} & -6+9.2^{2009} \\ -12+12.2^{2009} & 2+3.2^{2009} & -6+9.2^{2009} \end{pmatrix}$

$= -2^{2006}\left(\quad \right)$

Partie B: QCM: $\alpha = 1$		
Q1 a) et d)	Q2 a) et c)	Q3 b) et c)

Partie C: $\alpha = 0$ $M_0 = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix} = M_{\mathcal{B}}(f)$

Posons $f(e_1) = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = w_1$, $f(e_2) = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = w_2$

$f(e_3) = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = w_1$

1. M n'est pas inversible car $C_3 = C_1$.

2. comme ~~...~~ $f(e_1) = f(e_3)$

on a $\mathrm{Im}\, f = \mathrm{Vect}(f(e_1), f(e_2), f(e_3)) = \mathrm{Vect}(f(e_1), f(e_2))$

or w_1 et w_2 sont linéairement indépendants donc une base de $\mathrm{Im} f$ est (w_1, w_2).

Posons $w_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $w_3 \in \mathrm{Ker}\, f \iff f(w_3) = 0 \iff \begin{cases} x + z = 0 \\ -2x - y - 2z = 0 \\ -x - y - z = 0 \end{cases} \iff \begin{cases} x = -z \\ y = 0 \end{cases}$

soit $w_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\mathrm{Ker}\, f = \mathrm{Vect}(w_3)$.

Soit $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ $\alpha w_1 + \beta w_2 + \gamma w_3 = 0_{\mathbb{R}^3} \iff \begin{cases} \alpha - \gamma = 0 \\ -2\alpha - \beta = 0 \\ -\alpha - \beta + \gamma = 0 \end{cases} \iff \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases}$

(w_1, w_2, w_3) base de $\mathbb{R}^3 \Rightarrow \mathbb{R}^3 = \mathrm{Ker}\, f \oplus \mathrm{Im}\, f$